

Extremal Sasakian horizons

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Abstract

We point out a simple construction of an infinite class of Einstein near-horizon geometries in all odd dimensions greater than five. Cross-sections of the horizons are inhomogeneous Sasakian metrics (but not Einstein) on $S^3 \times S^2$ and more generally on Lens space bundles over any compact positive Kähler-Einstein manifold. They are all consistent with the known topology and symmetry constraints for asymptotically flat or globally AdS black holes.

Introduction: The classification of stationary black hole solutions in higher dimensional General Relativity is an important open problem [1], relevant to modern studies in quantum gravity. Unlike in four dimensions, black hole uniqueness is violated even for asymptotically flat vacuum spacetimes [2], however certain general results are known. Most notably the horizon topology is known to be positive Yamabe type [3] and the rigidity theorem guarantees that a rotating black hole has at least one commuting rotational isometry [4–6]. It is not known whether these conditions are sufficient for existence of a black hole solution: in fact very few explicit examples are known [2, 7, 8] (there are more solutions with disconnected horizons).

Extremal black holes are of particular importance in quantum gravity due to the fact they do not radiate. Hence a classification of such objects is particularly sought after. In fact it turns out that extremal black holes can be partly constrained by their near-horizon geometries. These can be studied independently of a full black hole solution thus providing a simplified setup in which to investigate issues such as horizon topology and symmetry of solutions [9, 10]. For simplicity we will focus our discussion on the vacuum Einstein equations (possibly allowing for a cosmological constant). In four and five dimensions near-horizon geometries with an appropriate number of commuting rotational symmetries have been classified [11–14]. In

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higher dimensions this has not yet been achieved, although some partial results are known [15–17]. In even dimensions certain classes of near-horizon solutions have been found which are different to the known (spherical) black hole solutions, but possess topology and symmetry consistent with the known general constraints for asymptotically flat and globally AdS black hole solutions [18]. It is the purpose of this note to present new examples, with non-trivial horizon topology, in all odd dimensions, again consistent with the topology and symmetry constraints for asymptotically flat and globally AdS black holes.

Motivated by the above considerations, the precise setup we will consider is as follows. Consider a D dimensional spacetime containing a (smooth) degenerate Killing horizon \mathcal{N} of a Killing vector field n , with a cross section H . In the neighbourhood of such a horizon, the spacetime metric written can be written in Gaussian null coordinates (see e.g. [19])

$$ds^2 = 2dv \left(dr + r h_A(r, x) dx^A + \frac{1}{2} r^2 F(r, x) dv \right) + g_{AB}(r, x) dx^A dx^B \quad (1)$$

where $n = \partial/\partial v$. The vector field $l = \partial/\partial r$ is tangent to the unique null geodesics that start on \mathcal{N} which satisfy $n \cdot l = 1$ and are orthogonal to tangent vectors on H . The coordinate r is an affine parameter and chosen such that $r = 0$ corresponds to \mathcal{N} , whereas (x^A) with $A = 1, \dots, D-2$, are coordinates on a cross section H of the horizon. We assume that H is a compact manifold (without a boundary). It is well known that for such spacetimes the Einstein equations restrict to a set of equations on H depending only on intrinsic data. We will consider vacuum spacetimes and allow for a cosmological constant $R_{\mu\nu} = \Lambda g_{\mu\nu}$, with particular interest in $\Lambda \leq 0$. In this case one can show that (see e.g. [10, 15])

$$R_{AB} = \frac{1}{2} h_A h_B - \nabla_{(A} h_{B)} + \Lambda g_{AB} \quad (2)$$

$$F = \frac{1}{2} h_A h^A - \frac{1}{2} \nabla_A h^A + \Lambda \quad (3)$$

where R_{AB} and ∇_A are the Ricci tensor and metric connection associated to the induced (Riemannian) metric on H defined by $g_{AB} \equiv g_{AB}(0, x)$, and $h_a \equiv h_a(0, x)$ and $F \equiv F(0, x)$. One can understand this result in terms of the so-called near-horizon limit. This is defined by replacing $r \rightarrow \epsilon r$ and $v \rightarrow v/\epsilon$ and then taking the limit $\epsilon \rightarrow 0$ [9, 10, 15]. The limit always exists for a (smooth) degenerate horizon resulting in the *near-horizon geometry*

$$ds_{NH}^2 = 2dv \left(dr + r h_A(x) dx^A + \frac{1}{2} r^2 F(x) dv \right) + g_{AB}(x) dx^A dx^B. \quad (4)$$

One can then check that ds_{NH}^2 satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$ if and only if (2) and (3) are satisfied. In this note we will present a simple set of solutions to the near-horizon equations (2) in all odd dimensions.

Construction of near-horizon geometries: We will make the following assumption: h^A is a Killing vector field. The horizon equation then reduces to

$$R_{AB} = \frac{1}{2} h_A h_B + \Lambda g_{AB}. \quad (5)$$

The contracted Bianchi identity then implies that $|h|^2 = h_A h^A$ is a constant (it follows that h^A is also tangent to geodesics). Therefore the scalar curvature $R = \frac{1}{2} |h|^2 + (D-2)\Lambda$ is a

constant, which we assume to be positive. Also note that equation (3) gives $F = \frac{1}{2}|h|^2 + \Lambda$ (and hence a constant). We deduce that for a non-static solution, h^A is a nowhere vanishing vector field on H . This is only possible if the Euler characteristic vanishes $\chi(H) = 0$, a constraint which is of course automatically satisfied in all odd dimensions¹. It is worth noting that any homogeneous near-horizon geometry necessarily has h Killing² so our considerations include such cases, however we will not assume homogeneity. Now there are two possibilities: either the orbits of h are closed or they are open. In the former case the orbit space is itself a compact manifold, which we refer to as the transverse space T . In the latter case the quotient space is only locally defined, but we still refer to it as the transverse space T .

It is convenient to use coordinates on H adapted to the Killing field h^A . Writing $(x^A) = (\bar{\psi}, y^a)$ where $a = 1, \dots, D-3$, so y^a are coordinates on the transverse space T , our near-horizon data reads

$$g_{AB}dx^A dx^B = (d\bar{\psi} + \hat{\sigma}_a dy^a)^2 + \hat{g}_{ab} dy^a dy^b, \quad h = k \frac{\partial}{\partial \bar{\psi}} \quad (6)$$

where $\hat{\sigma}_a$ and \hat{g}_{ab} are a 1-form and metric on T , and k is a constant so $h_A h^A = k^2$. This parameterisation is useful as $k = 0$ then corresponds to an Einstein manifold. Now define a 2-form on T by $\hat{\omega} = \frac{1}{2} d\hat{\sigma}$. The horizon equation (2) is then equivalent to a set of equations on the $D-3$ dimensional transverse space T :

$$\hat{R}_{ab} = 2\hat{\omega}_{ac}\hat{\omega}_b{}^c + \Lambda \hat{g}_{ab}, \quad \hat{\nabla}^a \hat{\omega}_{ab} = 0, \quad \hat{\omega}_{ab}\hat{\omega}^{ab} = \frac{k^2}{2} + \Lambda \quad (7)$$

where \hat{R}_{ab} and $\hat{\nabla}_a$ are the Ricci curvature and Levi-Civita connection associated to the transverse metric \hat{g}_{ab} . We will assume that the two-form $\hat{\omega}$ is non-trivial and hence require that $\frac{k^2}{2} + \Lambda > 0$. It is worth noting that the full near-horizon geometry can be written as

$$ds_{NH}^2 = \left(-\frac{k^2}{2} + \Lambda\right) r^2 dv^2 + 2dvdr + (d\bar{\psi} + \hat{\sigma}_a dy^a + kr dv)^2 + \hat{g}_{ab} dy^a dy^b \quad (8)$$

which is an H -bundle over AdS_2 (if $-\frac{k^2}{2} + \Lambda < 0$) preserving the full $SO(2,1)$ symmetry of AdS_2 , as is typically the case for near-horizon geometries [10, 16]. Hence all the solutions we discuss in this note also enjoy this near-horizon symmetry enhancement.

The set of equations (7) are difficult to solve in general³. However, let us now make a further assumption. Namely that the transverse space is $2n$ -dimensional with Kähler structure (T, \hat{g}, \hat{J}) and the Kähler form \hat{J} proportional to $\hat{\omega}$. This is equivalent to requiring our horizon to be locally Sasakian with h^A proportional to the Reeb vector. The horizon equations now simplify to

$$\hat{\omega}_{ab} = \sqrt{\frac{\frac{k^2}{2} + \Lambda}{2n}} \hat{J}_{ab}, \quad \hat{R}_{ab} = \left(\frac{k^2 + 2\Lambda(n+1)}{2n}\right) \hat{g}_{ab} \quad (9)$$

and thus the transverse space is also Einstein. It is worth pausing here to point out that for $n = 1$ this is the only solution to (7) and hence \hat{g} is locally isometric to the round

¹In even dimensions $\chi(H) = 0$ of course provides a non-trivial constraint on the topology. For example, it implies that H cannot be simply connected.

²By a homogeneous near-horizon geometry we mean (H, γ) is a homogeneous space such that the near-horizon data (h, F) is invariant under the isometries of γ . It easily follows that h^A is a Killing field.

³Indeed a similar set of equations arise in the classification of homogeneous metrics on circle bundles over Kähler-Einstein manifolds [20].

S^2 . The resulting horizon geometry is then a homogeneous metric on S^3 (or quotients) and the near-horizon geometry is locally isometric to that of the extremal Myers-Perry(-AdS) black hole with equal angular momenta. This therefore solves the classification problem for $D = 5$ homogeneous Einstein near-horizon geometries. From now on we will assume $n \geq 2$ so $D = 2n + 3 \geq 7$.

So far we have shown that given any Kähler-Einstein metric we can construct a solution to the horizon equation which is (locally) Sasakian. If T is a manifold then we may take any compact positive Kähler-Einstein manifold. For example, $T = \mathbb{CP}^n$ with the Fubini-Study metric gives a homogeneous horizon metric on S^{2n+1} , which corresponds to that of the extremal Myers-Perry black hole in odd dimensions with all angular momenta equal (see e.g. [16]). One could also choose $T = \mathbb{CP}^1 \times \mathbb{CP}^1$ which yields a homogeneous horizon metric on $T^{1,1} \cong S^3 \times S^2$. On the other hand, one may also choose a Kähler-Einstein base with no continuous isometries such as the higher del Pezzo surfaces, which results in a horizon geometry with a single rotational symmetry. These would saturate the lower bound for the rigidity theorem for black holes and are odd dimensional counterparts of the solutions in [18].

We may construct simple non-spherical horizon topologies by following the construction of the Sasaki-Einstein manifolds [23, 24]. Thus, we use the following Kähler metric on T :

$$\hat{g}_{ab} dy^a dy^b = \frac{x^{n-1} dx^2}{2P(x)} + \frac{2P(x)}{x^{n-1}} (d\bar{\phi} + \bar{\sigma})^2 + 2x\bar{g} \quad (10)$$

$$\hat{J} = d[x(d\bar{\phi} + \bar{\sigma})] \quad (11)$$

where $\text{Ric}(\bar{g}) = 2n\bar{g}$ and \bar{g} is itself a $2n - 2$ dimensional Kähler-Einstein metric on a base K with Kähler form $\bar{J} = \frac{1}{2}d\bar{\sigma}$. For $n = 2$ the only choice is $K = \mathbb{CP}^1 \cong S^2$ with round metric, which gives the cohomogeneity-1 Kähler metric with $SU(2) \times U(1)$ symmetry [25]. The Einstein condition $\text{Ric}(\hat{g}) = \hat{\lambda}\hat{g}$ is then simply:

$$P(x) = x^n - \frac{\hat{\lambda}x^{n+1}}{n+1} + c \quad (12)$$

where c is a constant. Therefore in order to get a solution to our horizon equation we simply require

$$\hat{\lambda} = \frac{k^2 + 2\Lambda(n+1)}{2n}. \quad (13)$$

To summarise our horizon metric takes the form

$$g_{AB} dx^A dx^B = [d\bar{\psi} + 2\omega_0 x(d\bar{\phi} + \bar{\sigma})]^2 + \frac{x^{n-1} dx^2}{2P(x)} + \frac{2P(x)}{x^{n-1}} (d\bar{\phi} + \bar{\sigma})^2 + 2x\bar{g} \quad (14)$$

where for convenience we have defined the positive constant

$$\omega_0 \equiv \sqrt{\frac{\frac{k^2}{2} + \Lambda}{2n}}. \quad (15)$$

The Sasaki-Einstein case is recovered by simply setting $k = 0$ (and furthermore one has to have $\Lambda = 2n > 0$). Thus we see that we have 1-parameter deformation of the Sasaki-Einstein spaces, given by the parameter k , which is contained entirely in the constant $\hat{\lambda}$. These deformations preserve the (local) Sasakian structure but are of course not Einstein. Crucially,

we may have $\Lambda \leq 0$, as required for their interpretation as horizons of asymptotically flat or AdS black holes.

In fact the horizon equation (5) is sometimes called the “ η -Einstein” condition and so the solutions we have obtained are examples of Sasaki η -Einstein manifolds [21].⁴ By convention, a Sasaki η -Einstein manifold of dimension $2n+1$ is normalised so that $\omega_0 = 1$. We will not fix this normalisation, but still refer to our class of horizon solutions as Sasakian, as one merely needs to rescale the metric by a constant factor to recover the standard definition.

Global analysis: We now turn to a global analysis of the above metric using the same strategy as in the Einstein case [23, 24] so we will be brief. We assume $\hat{\lambda} > 0$ which allows for $\Lambda \leq 0$ when $k \neq 0$. We want to find conditions such that our local metric (14) extends to a smooth metric on a compact manifold H . The local metric has potential singularities at $x = 0$ and the roots of $P(x)$. Compactness requires that we take $x_1 \leq x \leq x_2$ with $P(x) \geq 0$, where x_1, x_2 are two adjacent positive simple roots of $P(x)$. The only turning points of $P(x)$ are at $x = 0, n/\hat{\lambda}$ and thus its graph looks like a negative cubic. The necessary and sufficient conditions for existence of roots $0 < x_1 < x_2$ are:

$$c_* \equiv -\frac{1}{n+1} \left(\frac{n}{\hat{\lambda}} \right)^n < c < 0. \quad (16)$$

We now make a convenient local change of coordinates $(\bar{\psi}, \bar{\phi}) \rightarrow (\psi, \phi)$ defined by $\bar{\psi} = -2n\omega_0\psi/\hat{\lambda}$ and $\bar{\phi} = \phi/n + \psi$. The horizon metric can then be written as a local $U(1)$ fibration over a different base space B :

$$g = A(x)[d\psi + A_B]^2 + g_B, \quad A_B = \Omega(x)(d\phi + n\bar{\sigma}) \quad (17)$$

where $A(x) = \frac{2P(x)}{x^{n-1}} + 4\omega_0^2(x - \frac{n}{\hat{\lambda}})^2$ and $\Omega(x) = \frac{1}{nA(x)} \left[\frac{2P(x)}{x^{n-1}} + 4\omega_0^2x(x - \frac{n}{\hat{\lambda}}) \right]$ and

$$g_B = \frac{x^{n-1}dx^2}{2P(x)} + \frac{8\omega_0^2P(x)}{\hat{\lambda}^2A(x)x^{n-1}}(d\phi + n\bar{\sigma})^2 + 2x\bar{g}. \quad (18)$$

Now, using the explicit form for $P(x)$, it is easily seen that at any root $4\omega_0^2P'(x_i)^2 = \hat{\lambda}^2A(x_i)x_i^{2n-2}$. It can be checked that this implies one can simultaneously remove the conical singularities at $x = x_i$ in the (x, ϕ) part of the metric g_B . In particular, if we choose the co-ordinate ϕ to be periodic with period $\Delta\phi = 2\pi$, the (x, ϕ) part of the base metric g_B extends to a smooth metric on S^2 . We will make this choice, so that g_B gives a metric which is locally an S^2 -bundle over the Kähler-Einstein base (K, \bar{g}) . In order for this bundle to be globally defined on a compact total space H , one needs that the associated $U(1)$ -bundle is regular and K is a compact manifold. The normalised connection for this $U(1)$ -bundle is $A = d\phi + n\bar{\sigma}$ and so the periods for a basis of 2-cycles $\Sigma_I \subset K$ are

$$\frac{1}{2\pi} \int_{\Sigma_I} dA = \frac{n}{2\pi} \int_{\Sigma_I} 2\bar{J} = \int_{\Sigma_I} c_1(K) \equiv c_I \quad (19)$$

where we have used $\bar{\rho} = 2n\bar{J}$, where $\bar{\rho}$ is the Ricci form of K , and that the first Chern class $c_1(K) = [\bar{\rho}/2\pi]$. Because the latter is an integral class these periods $c_I \in \mathbb{Z}$ (Chern numbers)

⁴It is also worth emphasising that they have constant scalar curvature, and therefore amusingly they are also extremal in the sense used in geometry (i.e. they are critical points of $\int R^2$) [22].

and hence the bundle is automatically regular. Furthermore since $c_1(K) = -c_1(\mathcal{L})$, where \mathcal{L} is the canonical line bundle, we see that the $U(1)$ bundle is isomorphic to \mathcal{L}^* . To summarise, we have so far shown that g_B extends to a smooth metric on a base manifold B , which is an S^2 -bundle over K , if and only if $\Delta\phi = 2\pi$. Furthermore, this bundle is in fact the associated S^2 -bundle to the anti-canonical line bundle over K . For $n = 2$ we have $K \cong S^2$ and c_I are always even; hence B is a trivial S^2 -bundle over S^2 , so $B \cong S^2 \times S^2$.

The final part of the regularity analysis involves showing that our horizon metric g can be extended to a smooth metric on a $U(1)$ bundle over B , for a countable choice of values of c in the range (16). The 1-form A_B may be thought of as a $U(1)$ connection on B and it is readily verified that its curvature dA_B is globally defined on B . If we let the period of ψ be given by $\Delta\psi = 2\pi\ell$, then regularity of the $U(1)$ -bundle over B defined by this connection requires that the periods of $\frac{dA_B}{2\pi\ell}$ over a basis of 2-cycles for B are integers. A basis of 2-cycles for B is given by the S^2 fibre Σ (at a fixed base point on K) and by the two cycles $s\Sigma_I$, where the section $s : K \rightarrow B$ maps to the pole $x = x_2$ (say). Then we compute the Chern numbers

$$\frac{1}{2\pi\ell} \int_{\Sigma} dA_B = \frac{\Omega(x_2) - \Omega(x_1)}{\ell} \quad (20)$$

$$\frac{1}{2\pi\ell} \int_{s\Sigma_I} dA_B = \frac{\Omega(x_2)}{\ell} \int_{\Sigma_I} c_1(K) = \frac{\Omega(x_2)c_I}{\ell} \quad (21)$$

where c_I are the Chern numbers as above. We deduce that

$$\frac{\Omega(x_2) - \Omega(x_1)}{\ell} = p, \quad \frac{\Omega(x_2)}{\ell} = \frac{q}{I} \quad (22)$$

where p, q are non-zero integers and $I \equiv \gcd(c_I)$ (the Fano index of K). It follows that

$$\frac{\Omega(x_1)}{\Omega(x_2)} = 1 - \frac{Ip}{q}. \quad (23)$$

Hence existence of regular solutions reduces to solving the equation (23). Explicitly

$$R(c; \hat{\lambda}) \equiv \frac{\Omega(x_1)}{\Omega(x_2)} = \frac{x_1 \left(x_2 - \frac{n}{\hat{\lambda}} \right)}{x_2 \left(x_1 - \frac{n}{\hat{\lambda}} \right)} \quad (24)$$

defines a continuous function of c (for each $\hat{\lambda} > 0$), in the range (16). As $c \rightarrow 0^-$ we have $x_1 \rightarrow 0^+$, $x_2 \rightarrow (n+1)^-/\hat{\lambda}$ and $R(c; \hat{\lambda}) \rightarrow 0^-$. As $c \rightarrow (c_*)^+$ we have $x_1 \rightarrow (n/\hat{\lambda})^-$, $x_2 \rightarrow (n/\hat{\lambda})^+$ and $R(c; \hat{\lambda}) \rightarrow (-1)^+$. Therefore, for each $\hat{\lambda} > 0$, $R(c; \hat{\lambda})$ is a continuous function of $c \in (c_*, 0)$ such that in the limit $R(0, \hat{\lambda}) = 0$ and $R(c_*, \hat{\lambda}) = -1$. It immediately follows that for every pair of integers (p, q) such that $1 < Ip/q < 2$, there exist a solution to (23).

To summarise, we have shown that there exists a countably infinite number of smooth compact horizons (for each value of the continuous parameter k), labelled by pairs of integers (p, q) satisfying $1 < Ip/q < 2$. For $k = 0$ these reduce to the known Sasaki-Einstein manifolds [23, 24], whereas for $k \neq 0$ these give η -Einstein Sasakian manifolds of a similar nature (it is readily checked the Sasakian structure is globally defined). We will call these horizon manifolds $H^{p,q}$.

Topology: As we have seen $H^{p,q}$ is the total space of a $U(1)$ -bundle over B with Chern numbers (p, q) . The topology of these spaces is the same as the Sasaki-Einstein manifolds of [23, 24],

so again we will be brief. The base manifold B is itself an S^2 -bundle over a compact positive Kähler-Einstein manifold K . As K must be simply connected, it follows that B is as well. Furthermore, if p, q are co-prime, it follows that $H^{p,q}$ is also simply connected, which we assume henceforth. One can also show that B and $H^{p,q}$ are spin manifolds. For $n = 2$ we showed earlier $B \cong S^2 \times S^2$ and one can show that $H^{p,q} \cong S^3 \times S^2$. More generally, $H^{p,q}$ is a Lens space bundle over K . This can be seen as follows.

At a fixed base point K the 3d fibre is compact and has a $U(1)^2$ isometry generated by the 2π -normalised Killing fields $(\ell\partial_\psi, \partial_\phi)$. For each $i = 1, 2$, the canonically normalised Killing vector field $K_i = \Omega(x_i)\partial_\psi - \partial_\phi$ vanishes on exactly one codimension-2 submanifold given by $x = x_i$. These two sets of Killing fields are related by the matrix

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} \frac{q}{I} - p & -1 \\ \frac{q}{I} & -1 \end{pmatrix} \begin{pmatrix} \ell\partial_\psi \\ \partial_\phi \end{pmatrix}, \quad (25)$$

where we have used (22). The determinant of this matrix is p , which allows us to deduce that the fibre is a Lens space $S^3/\mathbb{Z}_p \cong L(p, 1)$.

Physical quantities: The area of the horizons $H^{p,q}$ is

$$A(H^{p,q}) = \frac{2^{n+2}\pi^2\ell\omega_0(x_2^n - x_1^n)\text{vol}(K)}{n\hat{\lambda}} \quad (26)$$

where $\text{vol}(K)$ is the volume of K . The Komar angular momentum with respect to a rotational Killing field m , for a spacetime containing a degenerate Killing horizon, can be evaluated as an integral over the horizon [16]

$$j[m] = \frac{1}{16\pi} \int_H \sqrt{g} h \cdot m. \quad (27)$$

For our horizons $H^{p,q}$ we get

$$j[\partial_\psi] = 0, \quad (28)$$

$$j[\partial_\phi] = \frac{2^{n-1}\pi\ell\omega_0^2 k(x_2^{n+1} - x_1^{n+1})\text{vol}(K)}{\hat{\lambda}n(n+1)}, \quad (29)$$

$$j[\bar{m}] = \frac{\int_K \bar{\sigma} \cdot \bar{m}}{\text{vol}(K)} j[\partial_\phi] \quad (30)$$

where \bar{m} is a rotational Killing field on K (should any exist). Somewhat surprisingly, we see there is no angular momentum in the direction of the $U(1)$ -fibre (ψ). In [18] it was shown that $\int_K \bar{\sigma} \cdot \bar{m} = 0$ for toric K , so in this case the spins associated to K also vanish.

Generalisations: Our construction admits a straightforward generalisation that gives horizon solutions with the same topology as the Sasaki-Einstein manifolds $L^{p,q,r}$, which in five dimensions are all diffeomorphic to $S^3 \times S^2$ [26, 27]. To generalise our five dimensional horizons, we need only replace (10, 11) with the following toric Kähler metric [27, 28]⁵

$$\hat{g}_{ab} dy^a dy^b = \frac{x-y}{2Q(y)} dy^2 + \frac{2Q(y)}{x-y} (d\hat{\phi} + x d\hat{\psi})^2 + \frac{x-y}{2P(x)} dx^2 + \frac{2P(x)}{x-y} (d\hat{\phi} + y d\hat{\psi})^2 \quad (31)$$

$$\hat{J} = d[(x+y)d\hat{\phi} + xy d\hat{\psi}] \quad (32)$$

⁵This in fact represents the most general orthotoric Kähler surface [28].

for which the Einstein condition $\text{Ric}(\hat{g}) = \hat{\lambda}\hat{g}$ is

$$Q(y) = \frac{\hat{\lambda}}{3}y(\alpha - y)(\alpha - \beta - y), \quad P(x) = -\frac{\hat{\lambda}}{3}x(\alpha - x)(\alpha - \beta - x) + c \quad (33)$$

where α, β, c are integration constants and without loss of generality we have used the translation freedom $(x, y, \hat{\phi}, \hat{\psi}) \mapsto (x + c, y + c, \hat{\phi} - c\hat{\psi}, \hat{\psi})$ to fix one of the roots of $Q(y)$ to zero. By a change of coordinates this base may be written in such a way that it contains the $n = 2$ cohomogeneity-1 metric (10). This is given by first assuming $\alpha \neq \beta$ and setting $y = (\alpha - \beta)\sin^2(\frac{\theta}{2})$, $\hat{\phi} \rightarrow \hat{\psi}/2$ and $\hat{\psi} \rightarrow (\beta\hat{\phi} - \alpha\hat{\psi})/[2\alpha(\alpha - \beta)]$. The base in the resulting coordinates then in fact allows one to set $\alpha = \beta = 3/\hat{\lambda}$ (the last equality is without loss of generality), which then reduces precisely to the cohomogeneity-1 case (10, 11) with $\bar{\phi} = (\hat{\phi} + \hat{\psi})/4$, $\bar{\sigma} = (1/2)\cos\theta d\bar{\chi}$ and $\bar{g} = (1/4)(d\theta^2 + \sin^2\theta d\bar{\chi}^2)$, where $\bar{\chi} = (\hat{\psi} - \hat{\phi})/2$. The global analysis of the resulting horizon manifold for $\alpha \neq \beta$ can be performed as in the Einstein case $L^{p,q,r}$ [26, 27], so we omit details. A similar construction can be performed to find cohomogeneity- n horizons with dimension $2n + 1$ with the same topology as the spaces $L^{p,q,r_1 \dots r_{n-1}}$ obtained in [26].

Summary: To summarise, we have given a simple construction of an infinite class of vacuum near-horizon geometries, allowing for a cosmological constant, in all odd dimensions $D = 2n+3$ greater than five. The horizon geometries are inhomogeneous Sasakian metrics on $S^3 \times S^2$ or more generally on $L(p, 1)$ -bundles over any compact positive Kähler-Einstein manifold K . These geometries have isometry group $G \cong U(1)^2 \times G_K$ where G_K is the isometry group of K . As a result, the Cartan subgroup of G is a subgroup of $U(1)^{n+1}$ (which is the Cartan subgroup of $SO(2n+2)$), as required for the horizon of an asymptotically flat or globally AdS black hole (for example with $K = \mathbb{CP}^{n-1}$ it is $U(1)^{n+1}$). Further, the horizon topologies are allowed by the known constraints for such black holes (i.e. positive Yamabe and cobordant to spheres). For a given K , they depend on two integers and a continuous parameter which corresponds to the Komar angular momentum of the horizon. It would of course be interesting to determine whether these are realised as the horizon geometries of yet to be found extremal black holes.

We have also found more general (non-Sasakian) horizon metrics in odd dimensions which include the examples given here. These will be presented elsewhere [29].

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